On Estimation of Variance Components in the Mixed-Effects Models for Longitudinal Data

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Abstract

For the mixed-effects models with two variance components which is often adopted for analyzing longitudinal data, we establish some necessary and sufficient condition for equality of the analysis of variance estimate and the spectral decomposition estimate of variance components. Thus when this condition is satisfied, both estimates share some statistical properties. Two practical examples satisfying the condition are given.

Keywords. Mixed-effects model; ANOVA estimate; Spectral decomposition estimate; Best linear unbiased estimate; Least squares estimate.

1. Introduction

In the last two decades the mixed-effects linear model has received considerable attention from both the theoretical and applied points of view, because of its extensive applications, for example, in analyzing longitudinal data problems arising in the biological health sciences, computer graphics and mechanic engineering and so on. For a comprehensive overview, we refer to Diggle, Liang and Zeger (1994) and Davidian and Giltinan (1996).

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In this paper we consider the following mixed-effects linear model with two variance components

$$y_{it} = x_{it}'\beta + u_i + \varepsilon_{it}, \qquad i = 1, \dots, N, \quad t = 1, \dots, T$$

where x_{it} is a $p \times 1$ vector of regressors, β is $p \times 1$ vector of unknown fixed effects, the random effect u_i and the error ε_{it} are assumed to follow the normal distribution with mean zero and variances σ_u^2 and σ_ε^2, respectively, and all u_i and ε_{it} are independent.

Writing this model in matrix form, we have

$$y = X\beta + Zu + \varepsilon, \tag{1.1}$$

where y and X are of dimensions $n \times 1$ and $n \times p$, respectively, where $n = NT$, $Z = I_N \otimes \mathbf{1}_T$, $\otimes$ denotes Kronecker product and $\mathbf{1}_T$ is a vector of ones of dimension $T, u = (u_1, u_2, \dots, u_N)'$ and $\varepsilon = (\varepsilon_{11}, \dots, \varepsilon_{1T}, \dots, \varepsilon_{N1}, \dots, \varepsilon_{NT})'$.

Dispersion matrix of observation vector y is given by

$$Cov(y) = \sigma_u^2 ZZ' + \sigma_\varepsilon^2 I, \tag{1.2}$$

where σ_u^2 and σ_ε^2 are unknown variance components

For the variance components, some popular estimates are analysis of variance estimate (ANOVAE), maximum likelihood estimate (MLE), restricted maximum likelihood estimate (RMLE) and minimum norm quadratic unbiased estimate (MINQUE), see, for example, Wang and Chow (1994). These estimates have some shortcomings in different extent, for example, ANOVAE and MINQUE can not guarantee the nonnegativity of estimates, see, for example, Kelly and Mathew (1993), however MINQUE , MLE and RMLE need to solve a system of non-linear equations, which usually do not have explicit solution and an iterative procedure is necessary, and MINQUE depends strongly on the initial guesses of the variance components, which has certain subjectivity, see Rao (1971). About statistical properties of these estimates, there are a few results in the literature up to now, so it is better to consider them as algorithms to produce some estimates.

Wang and Yin (2002) proposed a new method of simultaneously estimating fixed effects and variance components. The corresponding estimates are called as spectral decomposition estimate (SDE). Both the ANOVAE and SDE have their closed forms in all cases which can bring some convenience in further statistical analysis. The purpose of this paper is to establish some condition for the equality of the ANOVAE and SDE of variance components in model (1.1)

The structure of this paper is follow. In the next section, we introduce the ANOVAE and SDE of variance components in model (1.1). In Section 3 we obtain some necessary and sufficient condition for equality of the ANOVAE and the SDE. Finally, in section 4, two practical examples satisfying the condition are given.

Throughout this paper, A', $\mathbf{tr}(A)$, $\mathcal{M}(A)$, and A^- stand for the transpose, trace, column space and a generalized inverse of A, respectively. Further, denote $P_A = A(A^{'}A)^{-}A^{'}$, which is the orthogonal projector onto $\mathcal{M}(A)$, and $Q_A = I - P_A$.

2. Two estimates of variance components

For model (1.1), the ANOVAE of σ_ε^2 and σ_u^2 given by

$$\hat{\sigma}_\varepsilon^2 = \frac{1}{(n-r_0)} y'(I - P_{(X:Z)})y, \tag{2.1}$$

$$\hat{\sigma}_u^2 = \left[y'(P_{(X:Z)} - P_X)y - \frac{r_0 - rank(\ X)}{n - r_0} y'(I - P_{(X:Z)})y \right] /Ttr(Q_X P_Z). \tag{2.2}$$

where $r_0 = rank(X : Z)$. See, for example, Christensen (1996) p.281.

The SDE proposed by Wang and Yin (2002) based on the spectral decomposition of the covariance matrix, and then by using some appropriate linear transformation to obtain several new singular linear models. The feature of these models is that they have the same fixed effects as the original model, but their covariances matrices do not involve any unknown variance component except a factor (this factor is one of eigenvalues of the covariance matrix of original model). Using the unified theory of least squares (see, for example, Wang and Chow (1994) and Rao (1973)) for every new model, we obtain estimates for fixed effects and the eigenvalues. The eigenvalues of the covariance matrix of original model are linear functions of variance components, so by solving a system of linear equations, we can obtain the estimate of the variance components. The prominent feature of the new method is that for the fixed effects we can obtain several spectral decomposition estimates, they all have some good statistical properties, so we can make use of them to do further statistical inference such as testing of hypothesis, interval estimate and model diagnosis, and so on.

Follow the method described above, the SDE of σ_ε^2 and σ_u^2can be obtained as follow. Consider the spectral decomposition of covariance matrix

$$Cov(\ y) = \lambda P_Z + \sigma_\varepsilon^2 Q_Z,$$

where $\lambda = T\sigma_u^2 + \sigma_\varepsilon^2$. Premultiplying (1.1) by P_Z and Q_Z, respectively, gives

$$P_Z y = P_Z X\beta + \epsilon_1, \quad \epsilon_1 \sim N(0, \lambda P_Z), \tag{2.3}$$

$$Q_Z y = Q_Z X\beta + \epsilon_2, \quad \epsilon_2 \sim N(0, \sigma_\varepsilon^2 Q_Z). \tag{2.4}$$

Models (2.3) and (2.4) are two singular linear models. Based on the unified theory of least squares (Wang and Chow (1994), p.189), we can obtain the estimate of λ and σ_ε^2

$$\tilde{\sigma}_\varepsilon^2 = \frac{y'(Q_Z - Q_Z X(X'Q_Z X)^- X'Q_Z)y}{(rank(Q_Z) - rank(Q_Z X))}, \tag{2.5}$$

$$\tilde{\lambda} = \frac{y'(P_Z - P_Z X(X'P_Z X)^- X'P_Z)y}{(N - rank(P_Z X))}, \tag{2.6}$$

from $\tilde{\lambda} = T\tilde{\sigma}_u^2 + \tilde{\sigma}_\varepsilon^2$, we obtain the spectral estimator of σ_u^2

$$\tilde{\sigma}_u^2 = \left[\frac{y'(P_Z - P_Z X(X'P_Z X)^- X'P_Z)y}{N - rank(P_Z X)} - \frac{y'(Q_Z - Q_Z X(X'Q_Z X)^- X'Q_Z)y}{n - r_0}\right] /T. \tag{2.7}$$

In fact, the spectral estimates $\tilde{\sigma}_\varepsilon^2$ and $\tilde{\sigma}_u^2$ are obtained based on the Within residuals and the Between residuals provided by Swamy and Arora(1972), also see Baltagi (1995).

Since both the ANOVAE $(\hat{\sigma}_\varepsilon^2, \hat{\sigma}_u^2)$ and the SDE $(\tilde{\sigma}_\varepsilon^2, \tilde{\sigma}_u^2)$ are unbiased, thus in the rest of this paper we will focus on their variances to establish the condition of equality of the ANOVAE and SDE of the variance components $(\sigma_u^2, \sigma_\varepsilon^2)$.

3. Conditions for equality of two estimates

In order to prove our main results we need the following lemmas.

Lemma 3.1 For any matrices A and B with the same number of rows, denote $L = (A : B)$, then

$$P_{\mathbf{L}} = P_A + Q_A B(B'Q_A B)^- B'Q_A.$$

Note that $\mathcal{M}(L) = \mathcal{M}(A : Q_A B)$ and $A'Q_A B = 0$, it is easy to prove Lemma 3.1.

Lemma 3.2 For any matrices A and B with the same number of rows,

$$rank(A : B) = rank(A) + rank(B) - dim(\mathcal{M}(A) \cap \mathcal{M}(B)).$$

Since $\mathcal{M}(A : B) = \mathcal{M}(A) + \mathcal{M}(B)$ and $rank(A) = dim(\mathcal{M}(A))$, Lemma 3.2 follows directly from Theorem 2.1.1 of Wang and Chow (1994), p.11.

Lemma 3.3 Let $P = P_A P_B$, then

(a) P is an orthogonal projection matrix $\Leftrightarrow P_A P_B = P_B P_A$;

(b) if $P_A P_B = P_B P_A$, then P is the orthogonal projection matrix onto $\mathcal{M}(A) \cap \mathcal{M}(B)$.

The proof can be found in Wang and Chow(1994), p.37.

Lemma 3.4 The following three statements are equivalent:

(a) $P_A P_B = P_B P_A$,

(b) $\mathcal{M}(A) \cap \mathcal{M}(B) = \mathcal{M}(P_B A) = \mathcal{M}(P_A B)$,

(c) $rank(P_B A) = dim(\mathcal{M}(A) \cap \mathcal{M}(B))$.

Proof For any vector $c \in \mathcal{M}(A) \cap \mathcal{M}(B)$, there exist vectors α and γ, such that

$$\mathbf{c} = A\alpha = B\gamma,$$

$$P_B P_A c = P_B P_A A\alpha = P_B A\alpha = P_B B\gamma = B\gamma = \mathbf{c}.$$

Hence

$$\mathcal{M}(A) \cap \mathcal{M}(B) \subseteq \mathcal{M}(P_B P_A) \subseteq \mathcal{M}(P_B A), \tag{3.1}$$

from $(b) \Leftrightarrow (c)$ is proved.

It is now position to show $(a) \Leftrightarrow (b)$. If (a) holds, then

$$P_A(P_B A) = P_B P_A A = P_B A,$$

which means $\mathcal{M}(P_B A) \subseteq \mathcal{M}(A)$, but it is obvious that $\mathcal{M}(P_B A) \subseteq \mathcal{M}(B)$, thus

$$\mathcal{M}(P_B A) \subseteq \mathcal{M}(A) \cap \mathcal{M}(B),$$

from which and (3.1) we get the first equality in (b). The second equality in (b) follows from the symmetry of A and B in the statement .

Conversely, if (b) is true, then

$$\mathcal{M}(A) \cap \mathcal{M}(B) = \mathcal{M}(P_B P_A) = \mathcal{M}(P_B A), \tag{3.2}$$

which implies

$$P_A(P_B A) = P_B A,$$

and

$$P_B(P_A P_B) = P_A P_B. \tag{3.3}$$

By using (3.3), we have

$$P_B A(A' P_B A)^- A' P_B (P_A P_B) = P_B A(A' P_B A)^- A' P_B, \tag{3.4}$$

Note that

$$P_BA(A'P_BA)^-A'P_B(P_AP_B) = P_BA(A'P_BA)^-A'P_BA(A'A)^-AP_B = P_BP_AP_B, \tag{3.5}$$

combining (3.3), (3.4) with (3.5) gives

$$P_BA(A'P_BA)^-A'P_B = P_BP_AP_B = P_AP_B,$$

which shows that P_AP_B is the orthogonal projection matrix onto $\mathcal{M}(P_BA)$. By Lemma 3.3, (a) is proved. The proof of Lemma 3.4 is completed.

Theorem 3.1 For model (1.1), the SDE and ANOVAE of σ_ε^2 are equal in any case, that is

$$\hat{\sigma}_\varepsilon^2 = \tilde{\sigma}_\varepsilon^2.$$

Proof It follows from Lemma 3.1 that

$$Q_Z - Q_ZX(X'Q_ZX)^-X'Q_Z = I - (P_Z + Q_ZX(X'Q_ZX)^-X'Q_Z) = I - P_{(X:Z)}, \tag{3.6}$$

from which we can obtain

$$tr(Q_Z) - tr(Q_ZX(X'Q_ZX)^-X'Q_Z) = n - r_0.$$

Since both Q_Z and $Q_ZX(X'Q_ZX)^-X'Q_Z$ are idempotent, thus

$$rank(Q_Z) - rank(Q_ZX) = n - r_0. \tag{3.7}$$

From (2.1),(2.5), (3.6) and (3.7), the proof of Theorem 3.1 is completed.

In general, the SDE and ANOVAE of σ_u^2 are not equal. In what follows we will obtain a necessary and sufficient condition under which the two estimates of σ_u^2 are equal.

At first, we consider their variances. Denote

$$A_0 = \left[(P_{(X:Z)} - P_X) - \frac{(r_0 - rank(X))(I - P_{(X:Z)})}{n - r_0}\right] / Ttr(Q_XP_Z),$$

and

$$A_1 = \left[\frac{(P_Z - P_{ZX}(X'P_ZX)^-X'P_Z)}{N - rank(P_ZX)} - \frac{Q_Z - Q_ZX(X'Q_ZX)^-X'Q_Z}{n - r_0}\right] / T.$$

Note that $\tilde{\sigma}_u^2$ and $\hat{\sigma}_u^2$ are invariant estimators, let $\upsilon = y - X\beta$, then,

$$\hat{\sigma}_u^2 = \upsilon'A_0\upsilon, \quad \tilde{\sigma}_u^2 = \upsilon'A_1\upsilon.$$

By using Lemma 5.1.1 of Wang and Chow(1994) p.159, it is easy to show that if $\upsilon \sim N(0, V)$, then

$$Var(\upsilon' A \upsilon) = 2tr(AVAV).$$

In terms of this fact, we can prove that

$$Var(\hat{\sigma}_u^2) = 2\left[\sigma_\varepsilon^4 \frac{(n - rank(X))(r_0 - rank(X))}{T^2(tr(Q_X P_Z)^2(n - r_0)} + \sigma_\varepsilon^2 \sigma_u^2 \frac{2}{TtrQ_X P_Z)} + \sigma_u^4 \frac{tr(Q_X P_Z)^2}{(trQ_X P_Z)^2}\right],$$

and

$$Var(\tilde{\sigma}_u^2) = 2\left[\sigma_\varepsilon^4 \frac{(n + N - r_0 - rank(P_Z X))}{T^2(N - rank(P_Z X))(n - r_0)} + \sigma_\varepsilon^2 \sigma_u^2 \frac{2}{T(N - rank(P_Z X))} + \sigma_u^4 \frac{1}{N - rank(P_Z X)}\right],$$

It is now position to show the next theorem.

Theorem 3.2 $Var(\tilde{\sigma}_u^2) = Var(\hat{\sigma}_u^2) \Leftrightarrow P_X P_Z$ is symmetric matrix.

Proof For any $\sigma_u^2 > 0$ and $\sigma_\varepsilon^2 > 0$,

$$Var(\tilde{\sigma}_u^2) = Var(\hat{\sigma}_u^2)$$

if and only if

$$(a) \quad \frac{(n - rank(X))(r_0 - rank(X))}{(trQ_X P_Z)^2} = \frac{n + N - r_0 - rank(P_Z X)}{N - rank(P_Z X)},$$

(b) $tr(Q_X P_Z) = N - tr(P_X P_Z) = N - rank(P_Z X)$,

(c) $tr(Q_X P_Z)^2/(trQ_X P_Z)^2 = 1/(N - rank(P_Z X))$.

Denote $r_1 = dim(\mathcal{M}(X) \cap \mathcal{M}(Z))$. It follows from Lemma 3.2 and $rank(Z) = N$ that

$$r_0 = rank(X) + N - r_1.$$

Thus

$$n + N - r_0 - rank(P_Z X) = (n - rank(X)) + (r_1 - rank(P_Z X)),$$

It can be verified that $(a), (b)$ and (c) are equivalent to

$$tr(Q_X P_Z) = tr(Q_X P_Z)^2 = N - rank(P_Z X) = N - r_1. \tag{3.8}$$

Note that

$$Q_X P_Z = (Q_X P_Z)^2 \Leftrightarrow P_X P_Z = (P_X P_Z)^2,$$

thus

$$(3.7) \Leftrightarrow rank(P_Z X) = r_1,$$

by Lemma 3.4, the proof is completed.

Since both $\tilde{\sigma}_u^2$ and $\hat{\sigma}_u^2$ are unbiased estimators of σ_u^2, and if $P_X P_Z$ is symmetric matrix, they have the same variance , thus $P(\tilde{\sigma}_u^2 = \hat{\sigma}_u^2) = 1$. However, we have the following stronger result.

Theorem 3.3 If $P_X P_Z$ is symmetric matrix, then $\tilde{\sigma}_u^2 = \hat{\sigma}_u^2$.

Proof Suppose that $P_X P_Z$ is symmetric matrix, then by Lemma 3.2 and the proof of Lemma 3.4 and (3.6), we have

$$r_0 - rank(X) = N - r_1 = tr(Q_X P_Z) = N - rank(P_Z X),$$

$$P_Z X (X' P_Z X)^- X' P_Z = P_Z P_X = P_X P_Z = P_Z P_X P_Z.$$

Hence, to prove $\tilde{\sigma}_u^2 = \hat{\sigma}_u^2$, we only need to prove that the symmetry of $P_X P_Z$ implies

$$P_{(X:Z)} - P_X = P_Z - P_Z P_X P_Z. \tag{3.9}$$

In fact, from the symmetry of $P_X P_Z$ we can show that

$$(P_Z + P_X - P_Z P_X P_Z)^2 = P_Z + P_X - P_Z P_X P_Z,$$

and for any $c = Xa + Zb$,

$$(P_Z + P_X - P_Z P_X P_Z)c = c,$$

from which we obtain

$$\mathcal{M}(X : Z) \subseteq \mathcal{M}(P_Z + P_X - P_Z P_X P_Z) \subseteq \mathcal{M}(X : Z),$$

that is,

$$\mathcal{M}(X : Z) = \mathcal{M}(P_Z + P_X - P_Z P_X P_Z)$$

Hence

$$P_Z + P_X - P_Z P_X P_Z = P_{(X:Z)},$$

furthermore

$$P_{(X:Z)} - P_X = P_Z - P_Z P_X P_Z.$$

(3.9) is proved. The proof of Theorem 3.3 is completed.

4. Applications

In this section, two examples are given which the condition for equality of the ANOVAE and SDE is satisfied. Thus these estimates shares common statistical properties.

Example 1. Measuring shell velocities

Thompson(1963) discussed the problem of using several instruments to simultaneously measure the muzzle velocity of firing a random sample of shells from a manufacturer's stock. A suitable model for y_{ij}, the velocity of the ith shell as recorded by the j th measuring instrument, is

$$y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \tag{4.1}$$

$$i = 1, 2, \ldots, a, \qquad j = 1, 2, \ldots, b,$$

where α_i is the effect of the ith shell and β_j is the bias in the j th instrument. Since the shells fired are random sample of shells the α_i are random effects, and because the instruments used are the only instruments of interest,the β_j are fixed effects. So model (4.1) is a mixed model. See Searle (1971),p.381. Suppose that u and ε have the same distributions as in the model(1.1). we consider the estimate of variance components $(\sigma_u^2, \sigma_\varepsilon^2)$ in the model (4.1).

For the present model, the design matrices X and Z are given by

$$X = (1_a \otimes 1_b : 1_a \otimes I_b), \qquad Z = I_a \otimes 1_b.$$

It is easy to verify that

$$P_X P_Z = P_Z P_X = (\bar{J}_a \otimes I_b) \cdot (I_a \otimes J_b) = \bar{J}_a \otimes \bar{J}_b,$$

where $\bar{J}_n = 1_n\ 1_n'/n$.

Thus for the model (4.1), Theorems 3.1 and 3.3 hold, therefore the SDE and ANOVAE of variance components $(\sigma_u^2, \sigma_\varepsilon^2)$ are equal and given by

$$\begin{aligned}\tilde{\sigma}_\varepsilon^2 = \hat{\sigma}_\varepsilon^2 &= \frac{1}{(a-1)(b-1)} y'(I - \bar{J}_a) \otimes (I - \bar{J}_b) y \\ &= \frac{1}{(a-1)(b-1)} \Sigma\Sigma(y_{ij} - \bar{y}_{i.} - \bar{y}_{j.} - \bar{y}_{..})^2,\end{aligned}$$

$$\tilde{\sigma}_u^2 = \hat{\sigma}_u^2 = \frac{1}{b(a-1)} \left[y'((I - \bar{J}_a) \otimes \bar{J}_b) y - \frac{y'(I - \bar{J}_a) \otimes (I - \bar{J}_b) y}{b-1} \right]$$

$$= \frac{1}{b(a-1)}\left[\Sigma_i(\bar{y}_{i.} - \bar{y}_{..})^2 - (a-1)\tilde{\sigma}_{\varepsilon}^2\right],$$

where

$$\bar{y}_{..} = \frac{1}{ab}\Sigma\Sigma y_{ij}, \qquad \bar{y}_{i.} = \frac{1}{b}\Sigma_j y_{ij}, \quad i = 1, 2, \cdots, a.$$

Example 2. Fitting a circle to measured data

A circular feature in a mechanical object is one of the most basic geometric primitives.Its specification can be described easily by a center and a radius, due to imperfections introduced in manufacturing, machined parts will not be truly circular, the center and radius will in general specified by design engineers.

Wang and Lam (1997) present a mixed-effects model for circular measurements which took into consideration of the variability in center location of different machined parts. Let (x_{ij}, y_{ij}) be the jth measurement on circular machined part i, $i = 1, \ldots, m$, $j = 1, \cdots, n$,where m is the number of machined parts, and n is the number of measurements taken from the circumference of each machined part. The location of the center of part i is denoted by $(\xi + u_{1i}, \eta + u_{2i})'$, where $u_{11}, u_{12}, \cdots, u_{1m}$ and $u_{21}, u_{22}, \cdots, u_{2m}$ are assumed to be independently distributed $N(0, \sigma_0^2)$. Moreover, the radius of part i, ρ_i, is fixed but unknown. Let $\tau_{i(j)}$ be the angle of the jth measurement of part i. Because measurements are all taken with respect to some fixed but usually unknown direction, thus the angular difference between measurements, $\tau_{i(j+1)} - \tau_{i(j)}$, are assumed to be known and are all same for all machined parts. Thus it is assumed that $\tau_{i(j)} = \theta_{i0} + \theta_j$, $j = 1, 2, \cdots, n$, where θ_{i0} is fixed but unknown and θ_j is known. Define $\alpha_i = \rho_i cos\,\theta_{i0}$ and $\beta_i = \rho_i sin\,\theta_{i0}$. The measurement (x_{ij}, y_{ij})can be represented as follows

$$\begin{aligned} x_{ij} = &\quad \xi + \alpha_i\, cos\,\theta_j - \beta_i\, sin\,\theta_j + u_{1i} + \epsilon_{1ij}, \\ y_{ij} = &\quad \eta + \alpha_i\, sin\,\theta_j + \beta_i\, cos\,\theta_j + u_{2i} + \epsilon_{2ij}, \end{aligned} \tag{4.2}$$

where the disturbances ϵ_{1ij}, and $\epsilon_{2ij}^{'}$ are assumed to be independently distributed $N(0, \sigma^2)$ and they are also assumed to be independent of all u_{ji}.

Denote

$$z = (z_1, \cdots, z_m)^{'}, \quad z_i = (x_i', y_i')',$$

$$x_i = (x_{i1}, \cdots, x_{im})', \quad y_i = (y_{i1}, \cdots, y_{im})',$$

$$\gamma = (\xi, \eta, \alpha_1, \beta_1, \cdots, \alpha_m, \beta_m)^{'}, \quad u = (u_{11}, u_{21}, u_{12}, u_{22}, \cdots, u_{1m}, u_{2m})^{'}.$$

The model above can be expressed in the following matrix form

$$z = (1_m \otimes I_2 \otimes 1_n : I_{2m} \otimes \Phi)\gamma + (I_{2m} \otimes I_n)u + \epsilon,$$

where

$$\Phi = \begin{pmatrix} \phi_1 & -\phi_2 \\ \phi_2 & \phi_1 \end{pmatrix},$$

$$\phi_1 = (\cos\theta_1, \cdots, \cos\theta_n)^{'}, \quad \phi_2 = (\sin\theta_1, \cdots, \sin\theta_n)^{'}.$$

It can be verified that

$$Cov(\mathbf{z}) = \sigma^2 I_{2mn} + \sigma_0^2(I_{2m} \otimes \mathbf{1}_n\mathbf{1}_n^{'}).$$

For the present model, the design matrices X and Z are respectively

$$X = (1_m \otimes I_2 \otimes 1_n : I_{2m} \otimes \Phi), \qquad Z = I_{2m} \otimes I_n.$$

We note that the following conditions discussed by Wang and Lam (1997)

$$\begin{cases} \bar{c} = \frac{1}{n}\Sigma_{j=1}^n \cos\theta_j = 0, \\ \bar{s} = \frac{1}{n}\Sigma_{j=1}^n \sin\theta_j = 0, \end{cases} \tag{4.3}$$

implies that

$$P_X P_Z = P_Z P_X = \bar{J}_m \otimes I_2 \otimes \bar{J}_n.$$

Thus under the condition (4.3), the SDE and ANOVAE of variance components (σ_0^2, σ^2) are equal, and are given by

$$\tilde{\sigma}^2 = \hat{\sigma}^2 = \frac{1}{2m(n-2)}\Sigma\Sigma[\mathbf{x}_{ij} - \bar{\mathbf{x}}_{i.})^2 + (y_{ij} - \bar{y}_{i.})^2 - (\hat{\alpha}_i^2 + \hat{\beta}_i^2)],$$

$$\tilde{\sigma}_0^2 = \hat{\sigma}_0^2 = \frac{1}{2n(m-1)}\Sigma\Sigma[(\bar{\mathbf{x}}_{i.} - \bar{\mathbf{x}}_{..})^2 + (\ \bar{y}_{i.} - \ \bar{y}_{..})^2] - \frac{1}{n}\hat{\sigma}^2,$$

where

$$\hat{\alpha}_i = \frac{1}{n}\Sigma_j(x_{ij}\, cos\,\theta_j + y_{ij}\, sin\,\theta_j), \quad \hat{\beta}_i = \frac{1}{n}\Sigma_j(x_{ij}\, cos\,\theta_j - y_{ij}\, sin\,\theta_j).$$

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